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Introduction*

Policy makers, some argue, should not be concerned with the fact that income distribution becomes more uneven during the early stages of development. For one thing, there is the view that changing income distribution may be a consequence of the normal change in output composition during this period [13]. Rising per-capita product is generally accompanied by a rising share of non-agricultural output. If the non-agricultural sector is associated with an income distribution which is less even than that in agriculture, the change in sectoral weighting will cause the distribution of aggregate income to become more dispersed. In addition, it has been argued that in order for an economy to grow rapidly, income must become more unevenly distributed over time. Several economists have stressed the importance of an uneven distribution of income which favors entrepreneurs [10] and [14]. They contend that such a distribution, which is associated with industrialization, facilitates the mobilization of savings at low levels of per-capita product.

At an opposite pole to this view is the contention that government policies designed to re-distribute income early in the development process may increase both employment and output growth. The main point is that just as the distribution of income is affected by the composition of output, income distribution also influences the bill of goods which is demanded.

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As income becomes more evenly distributed, it is suggested that the demand for labor-intensive goods rises relative to demand for capital intensive goods. Moreover, this hypothesis has some empirical support which is brought out in Cline's study of two Latin American countries [7]. Thus, since economic systems are generally characterized by a surplus of labor in their early stages of development, policies which distribute income more evenly may increase aggregate output--as a consequence of their effect on factor opportunity cost.

According to this line of argument, other effects associated with a more even distribution of income would more than offset whatever dampening effect a lower level of private savings due to less inequality may have on growth. The level of total savings may be preserved by higher tax revenues and increased public savings. A more even distribution of income would cause the demand for imports to be reduced at a given level of per-capita income; therefore, it would reduce the impact of one of the main constraints on growth. Finally, the changes in output composition stemming from decreased income dispersion would reduce the aggregate capital-output ratio and the level of saving necessary to achieve a given growth rate.[1]

A third view of income distribution emphasizes that growth and equity may be conflicting objectives and that the choice of an approriate mix is an integral part of the problem of social welfare maximization.[2] Berry describes

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[1]Most of these ideas are expressed in [11, pp. 139-155].

[2]Berry points out that the possibility of a conflict between production maximization and distribution improvement rests upon the assumption that fiscal redistribution is relatively costly. In Berry's words, "...it is not possible to maximize production, forgetting about the distribution implicit in the particular way in which production is generated, and then redistribute income as seems appropriate after the fact of the production process"[4, p. 5].

this situation as follows:

> The relation between the two variables can be expressed in a 'possibilities curve,' where quantity of distribution is somehow measured on one axis and output growth on another; if the two are in conflict the 'possibilities curve' will have a negative slope. Since we may assume that a community indifference curve between the two would also have a negative slope, a tangency would, as in a regular indifference curve-production possibilities curve diagram, indicate the social optimum [4, p. 5].

The purpose of our paper is three-fold: (1) to isolate conditions under which any particular one of the three views just described may be consistent with an optimal income distribution path over time; (2) to demonstrate that a model which allows for the social costs of adjustment to reduced levels of private saving may have more than one turnpike; and (3) to show the sensitivity of the short-run and long-run growth paths to the weight given to dispersion in the welfare function.

We restrict ourselves to trading off only two effects of reduced income dispersion, those involving factor opportunity cost and savings. The others described above have not been adequately supported empirically.[1]

The problem consists of maximizing an integral of instantaneous welfare subject to two dynamic equations and initial and terminal conditions on the capital-labor ratio and the distribution of capital. The problem is designed

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[1]Cline [7] shows that in the two Latin American countries examined, Brazil and Mexico, the impact of improvements in income distribution on the demand for imports is negligible. On the other hand, in these two countries income equalization to the degree of equity of England produces a change in the composition of demand such that food and textiles make up a considerably larger share of total consumption. Given the high share of unprocessed food, these sectors are presumed to be relatively labor intensive, although Cline provides no conclusive evidence linking equalization to an increase in the aggregate employment capital ratio. Cline also demonstrates that in some countries the household savings rate is positively correlated with income.

to provide insight into optimal trajectory of the standard deviation of the distribttion of capital (including land, physical, and human capital). Once this trajectory has been determined, along with that of the income tax schedule and the capital labor ratio, inferences may be drawn about changes in the standard deviation of the distribution of real income.

In Section I, the model is described in detail. In Section II, the first-order conditions are derived and the turnpike properties of the system outlined. The last section gives the policy implications of the results.

I. The Model

Denote consumption per laborer of the jth household by c^j, capital per laborer by k^j, and non-capital income per laborer by ω^j. Then the function determining the consumption-labor ratio of the jth household may be written as

$$(1.1) \qquad c^j = \alpha_0 + \alpha_1 k^j + \omega^j$$

$$\alpha_0 \geq 0 \qquad 0 \leq \alpha_1 \leq 1$$

This function is consistent with a number of theories of consumption behavior. If it is assumed that the ratio of real cash balances to capital assets remains constant, then the relationship is similar to one proposed by Tobin [17] which makes consumption proportional to real wealth. If, on the other hand, individual households save in order to maintain a fixed ratio of capital assets to normal income, then, given no adjustment lag, α_1 may be interpreted as the product of the reciprocal of this ratio and the marginal propensity to consume out of normal income.

Non-capital income is untaxed and allocated completely to consumption expenditure. Since only the wages of unskilled labor are included in ω^j, this assumption of a unitary marginal propensity to consume may not be unrealistic.

By subtracting consumption per laborer from total household income per laborer, we obtain the function

$$(1.2) \qquad \begin{aligned} s^j &= y_D^j + \hat{\omega}^j - c^j \\ &= -\alpha_0 + y_D^j - \alpha_1 k^j \end{aligned}$$

where s^j is savings per laborer of the jth household and y_D^j is disposable capital income per laborer of the jth household. Denote aggregate domestic savings per laborer by s, disposable capital income per laborer by y_D, and capital intensity by k. Then taking the expected value of (1.2) yields the aggregate savings function

$$(1.3) \qquad s = -\alpha_0 + y_D - \alpha_1 k$$

For the sake of simplicity, net foreign capital inflow (which may be easily incorporated with the constant term α_0) is set equal to zero and the relationship

$$(1.4) \qquad i = s$$

where i is gross investment per laborer, is assumed to hold as an identity.

<u>The production-demand relationship</u>. Central to the model is a relationship giving gross domestic product per laborer as a function of the capital-labor ratio and the standard deviation of the distribution of capital. As Fisher [9] has shown, it is impossible to derive an aggregate production function when combining sectors which involve different commodities. Under certain conditions, however, functions relating total capital stock and employment to the factor-price ratio and gross output in each sector may be derived

by minimizing the cost functions of individual firms.

These functions make up what has been referred to as a "minimum requirements" isoquant and the basis for the production-demand relationship in our model.[1]

The function determining final demand per laborer in sector ℓ (z_ℓ) may be written as

$$(1.5) \qquad z_\ell = \phi_\ell(i, c^k, c^\omega, g, \sigma, \underline{P}) = \phi'_\ell(y_D, k, \hat{\omega}, g, \sigma, \underline{P})$$

where c^k is consumption per laborer financed out of capital income, c^ω is consumption per laborer financed out of the wages of unskilled labor, σ is the standard deviation of the distribution of k^j, and $\underline{P}$ is the vector of commodity shadow prices. This function combines several relationships, each of which corresponds to a component of final demand. To begin with, investment and government consumption demands by sector are assumed to be functions of total investment and total government consumption respectively. Private consumption demand for a commodity produced in sector ℓ is broken down into two components, c^ω_ℓ and c^k_ℓ, representing the different forms of consumption finance. These components are determined by the functions

$$(1.6) \qquad c^\omega_\ell = \theta_\ell(c^\omega, \underline{P})$$

$$(1.7) \qquad c^k_\ell = \mu_\ell(c^k, \sigma, \underline{P})$$

where $c^\omega = \hat{\omega}$ and $c^k = \alpha_0 + \alpha_1 k$. The wages of unskilled labor are equated

[1] "Minimum requirements" functions have been derived by Syrquin [16] in the case of a three-factor Cobb-Douglas production function. We assume that a linear input-output framework is applicable, which implies that the ratio of real value added to gross output remains constant in each sector. Consequently, the sector gross output vector may be included in the "minimum requirements" functions even though raw materials do not appear explicitly as a factor of production.

to the total consumption of this group. Since these payments are assumed to have a perfectly even distribution among households, no parameter of income dispersion enters (1.6).[1,2] Exports net of imports in each sector are functionally related to the shadow price of the commodity produced in the sector and the shadow price of foreign exchange. The latter variable is determined by requiring that the balance of payments identity be met (i.e., that the sum of the sectoral net exports equal zero).

Factor prices, the GDP-labor ratio and the employment rate enter (1.5) implicitly. This may be demonstrated by examining the expressions for $\hat{\omega}$, g, and $\underline{P}$. Values for $\hat{\omega}$ and g are given by the identities

(1.8) $$\hat{\omega} = \omega \cdot e$$

(1.9) $$g = y - y_D - \hat{\omega}$$

where ω is the wage rate of unskilled labor, e is the employment rate and y is the GDP-labor ratio.

The equation determining the vector of commodity shadow prices may be derived in the following manner: We assume that the form of the sectoral production functions is such that the value-added-gross output ratio in sector ℓ (v_ℓ), may be expressed as a function of the two factor prices[3]

(1.10) $$v_\ell = h_\ell(\omega, r)$$

where r is the rental return on capital.

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[1]One specific form of these functions may be derived as a simple extension of the consumption demand functions presented by Chenery and Raduchel [6].

[2]A more general assumption would be that the standard deviation of these payments is constant and that this distribution is independent of that of the k^j. This in no way affects the final results.

[3]Value added, acompiled by summing factor payments, may vary as a proportion of gross output, whereas real value added may not. Because of

Footnote #3 from page 7 cont.

the separability assumption implicit in the sectoral production functions, real value added may be set equal to the difference between gross output and intermediate inputs. See [15]. Factor payments are deflated by the GDP price index (P^{GDP}) and the ratio of value added (V) to real value added (V_i^R) in sector i is given by the identity $V_i/V_i^R = P_i^V/P^{GDP}$ where P_i^V is the value added price index for commodity i.

The shadow price of commodity ℓ is equal to the direct and indirect unit costs of producing commodity ℓ expressed in terms of the given set of factor prices. Thus,

$$(1.11) \qquad \underline{P} = \underline{\underline{a}}\,\underline{P} + \underline{v}$$

where an element $a_{m\ell}$ of the matrix a represents the amount of the mth commodity required to produce one unit of gross output in sector ℓ and $\underline{v}$ is the vector of $v_\ell^{\prime}$s. Re-arranging this expression, we obtain

$$(1.12) \qquad \underline{P} = (\underline{\underline{I}} - \underline{\underline{a}})^{-1}\underline{v} = (\underline{\underline{I}} - \underline{\underline{a}})^{-1}\underline{h}(\omega,\ r)$$

where $\underline{h}$ is the vector-valued function determining $\underline{v}$.

By substituting (1.8), (1.9), and (1.12) into (1.5), we obtain

$$(1.13) \qquad z_\ell = \phi_\ell\ (k,\ \omega,\ e,\ r,\ y,\ \sigma,\ y_D)$$

a linear transformation of these functions yields

$$(1.14) \qquad \underline{x} = (\underline{\underline{I}} - \underline{\underline{a}})^{-1}\underline{z}$$

$$= \phi^{\prime\ \prime\prime}(k,\ \omega,\ e,\ r,\ y,\ \sigma,\ y_D)$$

where $\underline{x}$ is the vector of sectoral gross outputs and $\underline{z}$ is the vector of sectoral final demands. The "minimum requirements" functions may be written as

$$(1.15) \qquad k = g\ (\frac{w}{r},\ \underline{x})$$

$$(1.16) \qquad e = \Psi\ (\frac{w}{r},\ \underline{x})$$

Substitution of (1.14) into these functions yields

(1.17) $k = g^{*} (w, r, k, e, \sigma, y, y_D)$

(1.18) $e = \Psi^{*} (w, r, k, e, \sigma, y, y_D)$

These functions are simplified in two ways. First, the labor supply function is assumed to take the form

(1.19) $e = e(w).$

$e_w > o \quad , \quad e_{ww} < o$

Given that this function is monotone, we may write

(1.20) $w = e^{-1} (e).$

This function is bounded in the following way

$\text{Limit}_{e \to i} \; w \to \infty$

$\text{Limit}_{e \to o} \; w \to \underline{w}$

where $\underline{w}$ is a politically determined minimum wage.

Since the sum of factor payments equals real GDP

(1.21) $r = (y - w \cdot e)/k$

By substituting this relationship into (1.17) and (1.18), the number of arguments upon which k and e depend may be reduced significantly.

The second way in which the functions determining k and e are simplified is by assuming that variations in the ratio of government consumption to gross investment (with the sum of i and g constant) do not affect k and e although they do affect output composition. The y_D variable enters equations (1.17) and (1.18) only through its effect on government consumption per laborer and gross investment per laborer. Now suppose that even though z_ℓ depends on g and i separately, K and e depend only upon the sum of g and i. Then the partial derivatives of (1.17) and (1.18) with respect to y_D will be zero, since

(1.22) $g + i = y - w - \alpha_o - \alpha_1 \cdot k$

The assumption that, cet. par., e and k are insensitive to variations in y_D is tenable. It may be argued that factor proportions in the sector producing capital goods are quite similar to those in the sector producing goods and services consumed by the government. For example, the average ratio of physical capital to output in both sectors may be relatively low particularly if a high weight is given to construction in the capital goods sector.

Under these assumptions, equations (1.17) and (1.18) may be re-written as

$$k = g^{*}(k, e, \sigma, y)$$

$$e = \Psi^{*}(k, e, \sigma, y)$$

Given that the system has a unique solution, these relationships may be used to determine e and y in terms of k and σ. The function determining GDP per laborer may be expressed as

(1.23) $$y = f(k, \sigma)$$

It can be shown that, under certain conditions, the derivatives of this function have the following properties

$$f_k > o \quad , \quad f_{\sigma k} > o$$

$$f_{kk} \leq o \quad , \quad f_{\sigma\sigma} \leq o$$

For any given value of k we must also require that (1.23) is maximized by a non-negative value of σ. A problem may arise in the case when $(\frac{\partial e}{\partial \sigma}) < o$ where the derivative is evaluated for values of σ greater than the optimum. If e approaches unity before σ reaches zero, we would expect this derivative to change sign. Under these conditions, the $(\frac{w}{r})$ ratio will approach infinity. If, however, the system is far from full employment as σ approaches zero, a negative value of σ may maximize (1.23) for a given value of k.[1]

[1]A theoretical argument may be made that both o and k should enter the labor supply function as well as w. This function would then be written as

$$e = e^{*}(k, \sigma, w)$$

The avoidance of a corner solution would require e^{*}_{σ} to be sufficiently large as o approaches zero such that $\text{Limit}_{\sigma \to o} (\frac{\partial e}{\partial \sigma})_{\Delta k = o} \geq o$

The Dynamics

The capital accumulation equation of the individual household takes the form

$$\dot{k}^j = -\alpha_o + y_D^j - (\alpha_1 + n + \delta)\, k^j \tag{1.24}$$

Denote taxes net of subsidies levied on the j-th household by NT^j. Now as

$$y_D^j = rk^j - NT^j \tag{1.25}$$

we have

$$\dot{k}^j = -\alpha_o + (rk^j - NT^j) - (\alpha_1 + n + \delta)\, k^j \tag{1.26}$$

One possible tax function is

$$NT^j = a_o'' + a_1'' \; (y_D^j - y_D) \tag{1.27}$$

which can be written as

$$NT^j = a_o'' + a_1'' \,.\, r \,.\, (k^j - k) \tag{1.28}$$

The coefficient a_o'' determines the revenue impact of the tax and the coefficient a_1' determines the re-distributive effect of the tax. Substituting (1.28) into (1.26) yields

$$\dot{k}^j = -\alpha_o + r\,\{k^j - \frac{a_e''}{r} - a_1''\,(k^j - k)\} - (\alpha_1 + n + \delta)\, k^j \tag{1.29}$$

Denote GDP by Y and aggregate capital stock by K. As

$$r \triangleq \frac{\partial Y}{\partial K} = f_k\,(k, \sigma) \tag{1.30}$$

we have

$$\dot{k}^j = -\alpha_o + f_k\,(k, \sigma)\,\{k^j - \frac{a_o''}{f_k} - a_1''\,(k^j - k)\} - (\alpha_1 + n + \delta)\, k^j \tag{1.31}$$

whence using

$$k \triangleq E\,(k^j) \tag{1.32}$$

$$\dot{k} \triangleq E\,\{\dot{k}^j\} \tag{1.33}$$

and

$$\dot{\sigma} \triangleq \frac{1}{\sigma}\; E\,\{\,(k^j - k)\,.\,\frac{d}{dt}\,(k^j - k)\,\} \tag{1.34}$$

we have

$$\dot{k} = -\,\alpha_o + f_k\,(k,\,\sigma)\;[\,k - a_o] - (\alpha_1 + n + \delta)k \tag{1.35}$$

$$\dot{\sigma} = f_k\,(k,\,\sigma)\;(1 - a_1)\;\sigma - (\alpha_1 + n + \delta)\;k\;. \tag{1.36}$$

We define the state variable ζ by the equation

$$1 - a_1 = \zeta$$

The control variables are given by the relationships

$$\dot{\zeta} = u_2 \tag{1.37}$$

$$a_o = u_1 = \frac{a_o''}{f_k(k,\,\sigma)} \tag{1.38}$$

Note

Note that u_1 is only a pseudo-control variable. In fact, it is the intercept of the net tax function (i.e., the expected net tax) which the government controls, not its capitalized value.[1]

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[1]It is assumed that the government controls a_o'' directly, whereas it influences only the change in the re-distribution coefficient a_1. This allows for the fact that the government regulates the level of gross investment by varying the government savings rate as well as net revenue. On the assumption that public capital formation benefits everyone equally, a coefficient $\hat{a}_o$, equal to a_o'' net of government investment, may be substituted for a_o'' .

The complete dynamics are then

(1.39) $$\dot{k} = -\alpha_o + f_k(k, \sigma)(k - u_1) - (n + \delta + \alpha_1)k$$

(1.40) $$\dot{\sigma} = \{f_k(k, \sigma) \cdot \zeta - n - \delta - \alpha_1\}\sigma$$

(1.41) $$\dot{\zeta} = u_2$$

<u>The Criterion Function</u>

Denote consumption (public and private) per laborer by c. Instantaneous welfare, U, is given by the function

(1.42) $$U = U(c, \sigma, u_1)$$

where

(1.43) $$c = f(k, \sigma) + \alpha_o + \alpha_1 k - f_k(k, \sigma)(k - u_1)$$

The inclusion of per capita consumption in (1.42) is certainly conventional. However, the same thing may not be said about the inclusion of u_1 and σ.

Arrow and Kurz [3] adduce a number of reasons why the ratio of government capital to labor should be included in the welfare function in addition to consumption per laborer. One of the most important of these is the effect of external economies arising from public capital which is not allowed for in the production-demand relationship. In general, it is clear that the benefits of government expenditure are not correctly valued in the national accounts; the specific direction of bias is uncertain.

The usual assumption about the savings impact of redistribution is not valid in the context of our model unless the benefits of government expenditure are understated. This assumption implies that a more even distribution of capital and income increases the social cost of saving. To reach this conclusion one must assume that the capitalized value of current government expenditure has a positive effect on welfare which goes beyond its contribution to total consumption, (i.e., $U_{u_1} > 0$). Otherwise, the government could compensate for whatever decrease in private savings resulted from an equalization of capital holdings at no social cost. This would be accomplished simply by reducing current government expenditures.

The inclusion of σ in the welfare function is highly debatable. It may not be feasible to determine the effect of changes in the distribution of income on social welfare. Difficulties arising from interpersonal comparisons of utility and the use of voting are outlined by Arrow in [2].

The criterion function itself may be written as

$$(1.44) \qquad \int_o^T e^{-\gamma t} U(c, u_1, \sigma)\, dt$$

where γ is the rate of social discount and T is the planning horizon.[1] Thus the paths of optimal capital accumulation and distribution are given by the solution(s) of the following optimal control problem:

$$(1.45) \qquad \underset{u_1,\, u_2}{\text{Max}} \int_o^T e^{-yt} U(c, u_1, \sigma)\, dt$$

where

$$(1.46) \qquad \dot{k} = -\alpha_o + f_k(k, \sigma)(k - u_1) - (\alpha_1 + n + \delta)k$$

$$(1.47) \qquad \dot{\sigma} = f_k(k, \sigma) \,.\, \delta - \alpha_1 - n - \delta\}\, \sigma$$

$$(1.48) \qquad \dot{\zeta} = u_2$$

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[1]The planning horizon, T, is assumed to be sufficiently large so that in the discussions of equilibrium solutions, the finiteness of T has little effect.

and the constraints

(1.49) $k \geq o, \sigma \geq o, \zeta \geq o$

(1.50) $$o \leq u_1 \leq \frac{-\alpha_o + \{f_k(k, \sigma) - \alpha_1\} k}{f_k(k, \sigma)}$$

(1.51) $A \leq u_2 \leq B$

with $A < o$, $B > o$

The upper bound on $u_{1\ z}$ reflects the fact the gross investment per laborer cannot be less than zero. The constraints on u_2 indicate that it is impossible to change the progressiveness of the tax instantaneously. As we shall show in the next section, these constraints are not germane to the rest of our discussion.

II. Equilibrium Growth Paths

We shall consider two cases. The first is based on the assumption that the partial derivatives of U with respect of $u_{1\ z}$ and σ are zero. The second concerns a more general situation in which U is a function of both u_1 and σ explicitly.

The optimal control problem referred to in the previous section is linear in the controls u_1 and u_2; thus the optimal policies will be of the "bang-singular-bang" type [5, pp. 261-65], i.e., the controls will move between their boundary values and an interior value(s) corresponding to the singular arc(s).

From the usual definition of equilibrium growth [1], we can see that the equilibrium solutions, if any, will be along the singular arcs; the constraints (1.49) and (1.50) are thus of no consequence as long as the equilibrium value of the capital intensity can be attained without u_1 going to zero. On the singular arc, when $H_u \equiv o$, the necessary conditions for optimality are derived from a Hamiltonian of the form

$$(2.1) \qquad H = e^{-\gamma t} [U(c) + \lambda_k \dot{k} + \lambda_\sigma \dot{\sigma} + \lambda_\zeta \dot{\zeta}]$$

Thus we have

$$(2.2) \qquad H_{u_1} = o = U_c \cdot f_k(k, \sigma) - \lambda_k f_k(k, \sigma)$$

$$(2.3) \qquad H_{u_2} = o = \lambda_\zeta$$

and

$$(2.4) \qquad -\dot{\lambda}_k = U_c \{- f_{kk} (k - u_1) + \alpha_1\} - \lambda_k \{n + \delta + \alpha_1 - f_k - f_{kk} (k - u_1)\} - \lambda_\sigma f_{kk} \sigma\zeta$$

$$(2.5) \qquad -\dot{\lambda}_\sigma = U_c \{f_\sigma - f_{k\sigma} (k - u_1)\} - \lambda_\sigma \{n + \delta + \gamma + \alpha_1 - f_k\zeta - f_{k\sigma}\sigma\zeta\} + \lambda_k f_{k\sigma} (k - u_1)$$

and

$$(2.6) \qquad -\dot{\lambda}_\zeta = -\gamma\lambda_\zeta + f_k \sigma \lambda \cdot \sigma$$

In addition, we have the dynamics which are given by equations (1.46), (1.47) and (1.48).

In equilibrium, $\dot{k} = o = \dot{\sigma} = \dot{\zeta} = \dot{\lambda}_k = \dot{\lambda}_\sigma = \dot{\lambda}_\zeta$,

This gives

$$(2.7) \qquad u_1 = k - \frac{(\alpha_1 + n + \delta)k + \alpha_o}{f_k(k, \sigma)}$$

(2.8) $$u_2 = o$$

(2.9) $$\zeta = \frac{n + \delta + \alpha_1}{f_k\ (k,\ \sigma)}$$

Also,

(2.10) $$\lambda_k = U_c$$

(2.11) $$\lambda_\zeta = o$$

Then from

(2.12) $$- \dot{\lambda}_\zeta = o = - \gamma\ \lambda_\zeta + \lambda_\sigma\ .\ \sigma . f_k\ (k,\ \sigma)$$

We have, assuming, $f_k\ (k,\ \sigma) \neq o$

(2.13) $$\lambda_\sigma = o$$

The two remaining variables are k, σ. These can be obtained from (2.4) and (2.5), giving

(2.14) $$f_k\ (k,\ \sigma) = n + \delta + \gamma$$

(2.15) $$f_\sigma\ (k,\ \sigma) = o$$

In equation (2.14), we have the familiar modified golden rule condition. The second signifies that the derivative of the production demand relationship with respect to σ is zero.

In order for a modified golden rule equilibrium to exist at the same time as the constraints on u_1 and ξ are satisfied, the condition

(2.16) $$\gamma > \alpha_1 + (\alpha_o/k.)$$

must hold.

This implies that both u_1 is greater than zero and that ζ is less than 1. If this condition is not met, the steady-state optimum can be attained only if "forced savings" (i.e., a negative u_1) and/or positive net foreign capital inflow are present. (Inclusion of the latter effect would cause α_o to be smaller.)[1]

It can be seen that, in this case, the equilibria are unique, provided that $f_k(k, \sigma)$ is concave in k and σ . However, if the function U(c) is altered so as to include the effects of u_1 and σ , i.e.,

(2.17) $$U \triangleq U(c, u_1, \sigma)$$

the following changes occur.

Equation (2.2) becomes

(2.2)' $$H_{u_1} = o = U_c f_k(k, \sigma) + U_{u_1} - \lambda_k f_k(k, \sigma)$$

and (2.5) becomes

(2.5)' $$\begin{aligned} -\dot{\lambda}_\sigma = & U_c \{f_\sigma - f_{k\sigma}(k - u_1)\} + U_\sigma \\ & - \lambda_\sigma \{n + \delta + r + \alpha_1 - f_k\zeta - f_{k\sigma} \cdot \sigma . \zeta\} \\ & + \lambda_k . f_{k\sigma}(k - u_1) \end{aligned}$$

The equilibrium conditions are now

(2.7)' $$u_1 = k . [1 - \frac{n + \delta + \alpha_1}{f_k(k, \sigma)}] + \frac{\alpha_o}{f_k(k, \sigma)}$$

(2.8)' $$u_2 = o.$$

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[1] Recall the regularity condition that for any given value of K the value of σ which maximizes $f(k,\sigma)$ is non-negative. This condition implies that $f_\sigma(k, \sigma) = o$ at $\sigma \geq o$.

(2.9)' $$\zeta = \frac{n + \delta + \alpha_1}{f_k\ (k, \sigma)}$$

(2.10)' $$\lambda_k = U_c + \frac{U_{u_1}}{f_k\ (k, \sigma)}$$

(2.11)' $$\lambda_\zeta = o$$

(2.13)' $$\lambda_\sigma = o$$

(2.14)' $$f_k\ (k, \sigma) = n + \delta + \gamma + \frac{U_{u_1}}{U_c}\ \{\frac{(n + \delta + \gamma + \alpha_1}{f_k\ (k, \sigma)} - \frac{(n + \delta)}{f_k^2(k,\sigma)}\ \cdot\ k\ \cdot\ f_{kk} - 1\}$$

(2.15)' $$f_\sigma\ (k, \sigma) = -\frac{U_{u_1}}{U_c}\ \{\frac{f_{k\sigma}\ (k, \sigma)\ .\ \{\ (n + \delta + \alpha_1)\ k - \alpha_o\}}{f_k^2\ (k, \sigma)}\} = \frac{U_\sigma}{U_c}$$

Clearly, more than one combination of k and σ may satisfy these conditions.

Aside from the possibility of multiple turnpikes, these conditions differ from the ones obtained earlier in other respects. From (2.10), observe that the modified golden rule condition never holds. Since $f_{kk} < o$, in equilibrium the inequality

$$f_k\ (k, \sigma) > \ n + \delta + \gamma$$

must be satisfied provided that $U_{u_1} > o$. Again in contrast to the previous results, the partial derivative of the production demand relationship with respect to σ need not equal zero along the turnpike.

This is true simply because the effect of a change in σ on welfare goes beyond its effect on consumption per laborer. Condition (2.15)' states that the increase (decrease) in welfare due to an increase (decrease) in consumption per laborer, $f_\sigma (k, \sigma) \cdot U_c$, must equal the sum of the two other welfare effects resulting from a change in σ. The first is the partial derivative of the welfare with respect to σ multiplied by -1. The second is the change in welfare stemming from the change in the level of u_1 necessary to maintain the equilibrium capital intensity; this value is given by the term

$$- U_{u_1} \left\{ \frac{f_{k\sigma} (k, \sigma) \, [(n + \delta + \alpha_1) k - \alpha_o]}{f_k^2 (k, \sigma)} \right\}$$

A relevant example is the case where $f_\sigma < o$. Here the social value of the rise in consumption per laborer due to a decline in σ must equal the social value of the loss in capitalized government expenditures net of the direct welfare gain.

Although the equilibrium value of U_σ does affect σ, it has no effect on ζ or u_1 since $f_k (k, \sigma)$ is independent of U_σ. Thus, the coefficients of the net tax equation may be determined in the case of the long-run optimal trajectory without knowing effect of changes in σ on social welfare.

III. Conclusion

The standard deviation of the distribution of disposable income per laborer (σ_y), in a steady-state equilibrium is a linear function of σ. For, from (1.25) we see that the total disposable income (which is the sum of income from capital y_D, and wage income w) is given by

$$\text{Total disposable income} = r \cdot k + w - \text{Expected net tax.}$$

Then, noting the fact that wage income w, is assumed to be uniformly distributed, we see that

$$\sigma_y = (1 - a_1)\ \sigma\ f_k = \zeta\ .\ \sigma\ .\ f_k$$

Then from (2.9)

$$\sigma_y = (n + \delta + \alpha_1)\ .\ \sigma$$

The optimal trajectory for k, σ, and σ_y depends on where the initial values for these two variables lie relative to the steady-state optimum values given by (2.14)' and (2.15)'. In the case where one or more turnpikes exist and the initial condition for σ lies below all the steady-state optimum values, it will be optimal to increase capital dispersion initially. If the terminal condition on σ requires that it lie below the turnpike values, then the optimal path for σ will arch toward the turnpike value and then eventually arch down to the terminal value. Assuming that the initial and terminal values for σ satisfy two different steady-state equilibrium conditions, σ_y will follow a similar pattern. This trajectory is comparable to that observed historically by Kuznets [12].

The view that income equalization and economic growth are consistent objectives holds true when, given that U_σ is zero everywhere, welfare may be maximized by increasing k and simultaneously lowering σ. Under these circumstances the increment in welfare due to the expansionary effect which a decline in dispersion has on GDP per laborer must outweigh the social cost of the decline in capitalized government expenditures necessary to keep k constant. Since the initial value of k is below the optimum, we are assured that this condition will not be met by a downward adjustment in both k and σ which could lead to a decrease in GPP per laborer. However, in many cases, finding the optimal trajectory for σ will not be so simple. In this paper we have indicated that when U_{u_1} is non-zero, multiple turnpikes may exist; thus, the initial value of σ will depend upon a variety of factors such as the proximity of the initial value to a given steady-state level, the stability of the various solutions and the initial condition on k.

One interesting situation that may arise in the multiple-turnpike case appears to have a direct bearing on the effect of postponing income redistribution in developing countries. Consider the case when $U_{u1} \neq 0$ and the initial values of k and σ lie between two turnpikes; along one k and σ are larger than along the other. There is evidence to suggest that in developing countries income becomes more unevenly distributed as per capita consumption and the capital intensity rises. Thus by not deviating from the normal pattern of increasing capital accumulation at the expense of further dispersion, policy makers may reach a stage beyond which income equalization may no longer be optimal. In other words, postponing re-distribution may lead the society to a set of initial conditions from which it is optimal to arch towards the turnpike with the higher value of σ, whereas a trajectory with a lower value of income dispersion may have been possible in the past.

One final point that should be made is that the form of the welfare function may affect the equilibrium values of σ without affecting the optimal trajectory of ζ (which is equal to one minus the re-distribution coefficient, a_1). Thus when the initial value of the re-distribution coefficient lies below the relevant turnpike, it is optimal to set a_1 at its upper bound until the turnpike is reached, and conversely for the case when the initial value lies above. Moreover, the turnpike value of a_1 is independent of the form of U_σ.

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