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THE COBWEB PHENOMENON IN SUBSISTENCE AGRICULTURE: A THEORETICAL ANALYSIS

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Peasants in underdeveloped areas consume a large part of their own output. Since a change in agricultural prices affects their income and, therefore, the consumption of their own goods, an analysis of the dynamic behavior of prices should take this factor into account. What follows incorporates peasants' income in a dynamic analysis of agricultural prices with the simple cobweb model as the starting point. It is shown that the introduction of an income term completely alters the stability condition. In this simple case, if the system has any equilibrium point at all, it will, in general, have two points of equilibrium; one of these is stable while the other is unstable.

The shortcomings of the simple cobweb model or its more elaborate version with adaptive expectations are well known by now.[1] It has been pointed out that price expectations in such models are irrational because ". . . the time path of actual and

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[1]See M. Ezekiel, "The Cobweb Theorem," Quarterly Journal of Economics, Vol. 52, and M. Nerlove, "Adaptive Expectation and Cobweb Phenomena, "Quarterly Journal of Economics, Vol. 73, No. 2. For critical appraisals of the cobweb models see J.F. Muth "Rational Expectations and the Theory of Price Movements," Econometrica, July 1961, and Edwin S. Mills, "The Use of Adaptive Expectations in Stability Analysis," Quarterly Journal of Economics, Vol. 75, No. 2.

expected prices are related in such a way that it is implausible to assume that decision makers continue to form expectations in the way postulated.[2] Furthermore, the nonstochastic nature of the model limits its application to real life problems. The same criticism applies to the model discussed in this paper. Our use of the cobweb model is not based on any supposed realism. Rather, we intend to use its simplicity to illustrate how the introduction of an income term can change the stability of a system.

Let us consider an economy consisting of two sectors: an urban sector and a rural sector. There is only one agricultural good produced. We further assume that the rural sector's demand for industrial goods is a small proportion of the total demand for such goods. Therefore, we can neglect its effect on industrial prices or incomes and take these as exogenously determined. Let

Y_t = rural sector's income in period t

P_t = price of the agricultural good in period t

S_t = production of the agricultural good in period t

D_{ut} = urban sector's demand for the agricultural good in period t

D_{rt} = rural sector's demand for the agricultural good in period t

By definition, $Y_t = K + S_t P_t$

where K is a nonnegative constant which we may consider as income from other sources.

We make the following behavioral assumptions:

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[2] E.S. Mills, Op.cit., p. 330.

(i) $S_t = S\ (P_{t-1})$, i.e., production of the agricultural good in period t depends on its price in period t-1.

(ii) $D_{ut} = D_u\ (P_t)$, the urban sector's demand for the agricultural good is a function of its price. This follows from our assumption that the urban sector's income is constant.

(iii) $D_{rt} = D_r\ (Y_t, P_t)$, the rural sector's demand for the agricultural good is a function of its price and the income of the rural sector in period t.

(iv) There are no inventory changes and the market is cleared in each period by price changes which will equate demand and supply.

From our last assumption about market clearance we have the following equation:

$$D_u(P_t) + D_r(Y_t, P_t) = S(P_{t-1})$$

Substituting for Y

$$D_u(P_t) + D_r(K + P_t S(P_{t-1}), P_t) = S(P_{t-1}) \qquad (1)$$

Equation (1) is a difference equation in P, whose solution gives the dynamic behavior of prices. In order to solve this equation, the form of the various functions must be specified. For simplicity we assume these functions are linear in form. (If the functions are nonlinear, it may still be possible to examine the local stability of the system by considering linear approximations near a point of equilibrium.) Accordingly, we let:

$$D_u(P_t) = a + b\ P_t$$

$$D_r(Y_t, P_t) = c + dY_t + eP_t$$

$$S(P_{t-1}) = g + hP_{t-1}$$

In addition, we make the following assumptions about the signs of our parameters a, b, c, d, e, g and h:

$a > 0$ i.e., at $P = 0$ a positive quantity is consumed

$b < 0$ our good is not inferior and the demand curve has the usual negative slope

$c > 0$ this implies the rural sector has an income elasticity of less than one for the agricultural good

$d > 0$ i.e., the good in question is not an inferior good for the rural sector

$e < 0$ again the usual assumption about the slope of the demand curve

$g \geqq 0$ this implies that the supply function is inelastic. But the sign of g is crucial only insofar as we require a positive quantity to be produced at the equilibrium price

$h > 0$ there is a positive production response to price.

Equation (1) now reads:

$$a + b P_t + c + d [K + (g + h P_{t-1}) P_t] + eP_t = g + h P_{t-1} \qquad (2)$$

It is worth noting that if d = 0, i.e., if there is no income effect, equation (2) reduces to

$$a + (b+e) P_t = g + b P_{t-1}$$

which is the familiar equation for the simple cobweb model.

Solving for P_t in equation (2), we get

$$p_t = \frac{-(a + c - g + dK) + h P_{t-1}}{(b + dg + e) + dh P_{t-1}} \qquad (3)$$

For simplification, we make the following substitutions:

$$-(a + e - g + dK) = \alpha$$
$$h = \beta$$
$$b + dg + e = \gamma$$
$$dh = \delta$$

Equation (3) therefore becomes

$$P_t = \frac{\alpha + \beta P_{t-1}}{\gamma + \delta P_{t-1}} \tag{4}$$

Our assumptions about the signs of the original parameters tell us that

$$\alpha \gtreqless 0$$

$$\beta > 0$$

$$\gamma \gtreqless 0$$

$$\delta > 0$$

We shall show presently that the stability of the system depends solely on the signs of β and δ.

Equation (4) is a first order nonlinear difference equation which can be solved explicitly for P_t. But to examine the stability of the system it is sufficient to look at the phase diagram which depicts the relationship between P_t and P_{t-1} which in our case (i.e., a first order difference equation) is the difference equation itself. A point of equilibrium is shown by the intersection of the phase diagram and the 45° line through the origin.

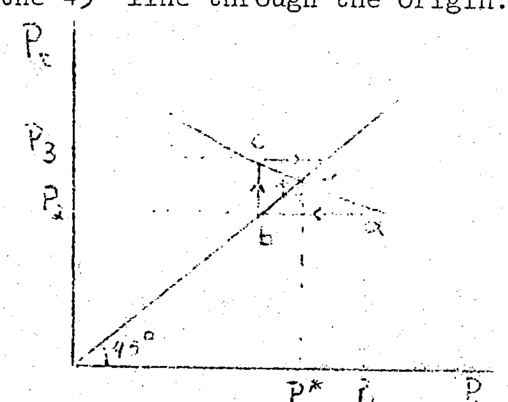

Figure 1

Figure 1 shows the phase diagram of a hypothetical system. If the system starts with $P_{t-1} = P^*$, then $P_t = P^*$ and P_{t+1} will be P^*, and so on. There is no tendency to move away from this point which is, therefore, a point of equilibrium. For the existence of an equilibrium point we require that the 45^o line through the origin intersect the phase diagram at least once. The stability of the equilibrium depends on the absolute value of $\frac{dP_t}{dP_{t-1}}$ near the point of intersection; the equilibrium is unstable if

$$\left| \frac{dP_t}{dP_{t-1}} \right| > 1$$

and it is stable if

$$\left| \frac{dP_t}{dP_{t-1}} \right| < 1 .$$

The time path of the variable in question can be found by the following method which is similar to the graphical method used in the familiar cobweb or the reaction curve diagrams.[3]

Suppose, in period t, the system starts from an arbitrary value of the variable equal to P_1. In period 2, P can be found directly from the graph of P_t and P_{t-1}. To find P in period 3 we have to put P_{t-1} equal to P_2 and read P_3 from the graph. This can be done by drawing ab parallel to the P_{t-1} axis and bc parallel to the P_t axis.

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[3]For a more complete discussion of phase diagrams see W. J. Baumol, "Topology of Second Order Linear Difference Equations With Constant Coefficient," Econometrica, Vol. 26.

Each subsequent period's P can be found by repeating this procedure. A typical path is shown in the figure.

We now examine the phase diagram of our model which is given in equation (4). If the equilibrium point is denoted by P*, we require the equation

$$P^* = \frac{\alpha + \beta P^*}{\gamma + \delta P^*}$$

to have at least one real root. Solving for P* we get

$$\delta P^{*2} + (\gamma - \beta) P^* - \alpha = 0$$

$$\therefore P^* = \frac{-(\gamma - \beta) \pm \sqrt{(\gamma - \beta)^2 + 4\alpha\delta}}{2\delta}$$

P* is real as long as $(\gamma - \beta)^2 + 4\alpha\delta \geq 0$. For the rest of our analysis we assume this condition is satisfied. Only the positive roots are of interest to us because we cannot have negative prices.

Equation (4) is the equation of a rectangular hyperbola with asymptotes $\frac{\beta}{\delta}$ and $-\frac{\gamma}{\delta}$. It is well known that a rectangular hyperbola is symmetrical about the 45° line going through its center, which in this case is $(-\gamma/\delta, \beta/\delta)$. In this phase diagram

$$\frac{dP_t}{dP_{t-1}} = \frac{\gamma\beta - \alpha\delta}{(\gamma + \delta P_{t-1})^2}$$

Since the denominator is always positive, $\frac{dP_t}{dP_{t-1}}$ has the same sign as $\gamma\beta - \alpha\delta$, which can be either positive or negative, as shown in Figures 2 and 3, respectively.

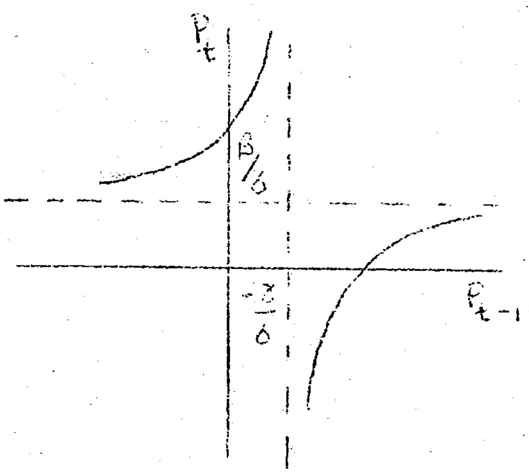


Figure 2

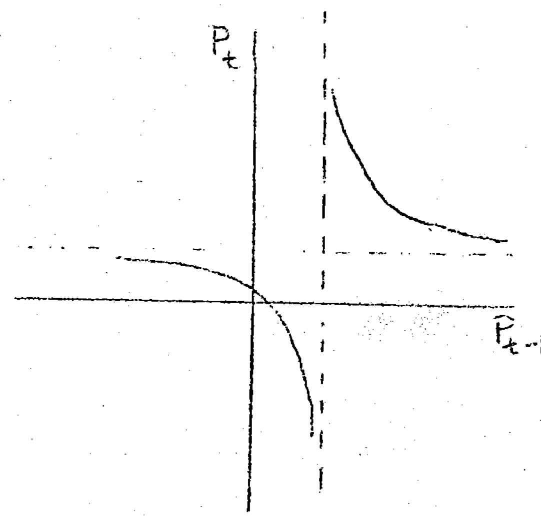


Figure 3

The slope of a rectangular hyperbola changes monotonically between 0 and ∞ or 0 and $-\infty$ (depending on the sign of the slope) on either side of the vertical asymptote. Further, its absolute value is one at the intersection of the curve with the 45° line through its center.

The values of α, β, γ and δ determine the position of the phase line with respect to the 45° line through the origin.

But the phase line always remains a rectangular hyperbola since by assumption δ is greater than zero. Different values of the parameter only change the position of the asymptotes. The asymptote parallel to the P_{t-1} axis ($P_t = \beta/\delta$) will always be positive, because both β and δ are positive by assumption. There are a number of cases to consider and we shall show that there is one point of stable equilibrium in each case.

Since the model describes an economic system in which we rule out negative prices and quantities, we have to specify another rule to take care of this. Therefore, we assume

$$P_t = \text{Max} \left[\frac{\alpha + \beta P_{t-1}}{\gamma + \delta P_{t-1}}, 0\right] \tag{5}$$

i.e., prices cannot become negative.

Case (1). $\gamma > 0$

This implies that the asymptote parallel to the P_t axis is negative. Diagrams 4 and 5 show the two possible cases with $\frac{dP_t}{dP_{t-1}}$ positive and negative respectively. In both cases, the intersection of the 45^o line and the phase line in the N.E. quadrant is stable, as is easily verifiable by tracing a time path in the diagram.

Whenever P_{t-1} is the interval $(\frac{-\gamma}{\delta}, \frac{-\alpha}{\beta})$, the following period's price will be zero (see equation (5)), which implies that P_t will eventually approach P^*_1.

Case (3). $$\frac{dP_t}{dP_{t-1}} < 0 \quad \text{and} \quad \frac{-\gamma}{\delta} < \frac{\beta}{\delta}$$

In this case, P^*_2 is the stable equilibrium point and P^*_1 is the unstable equilibrium (see Figure 7).

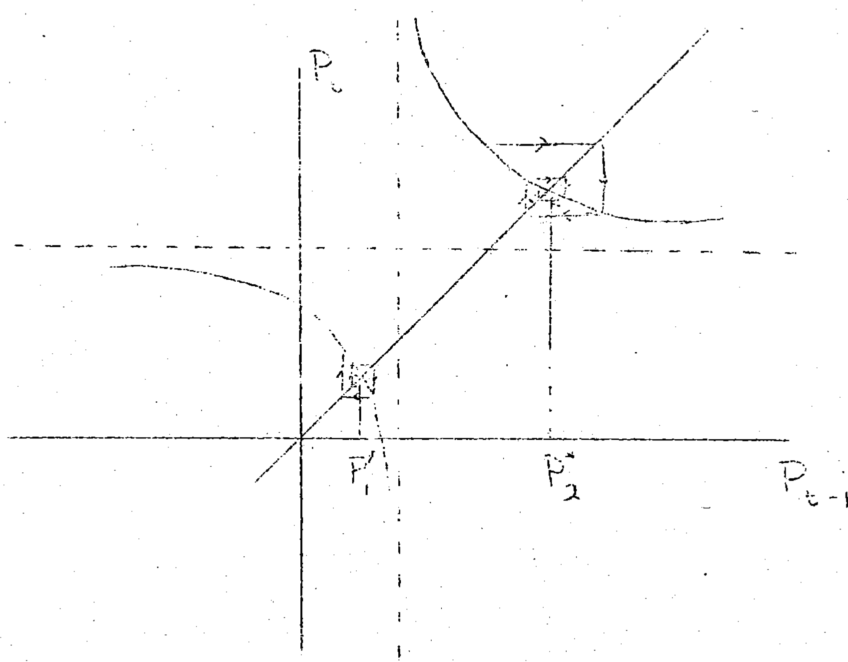
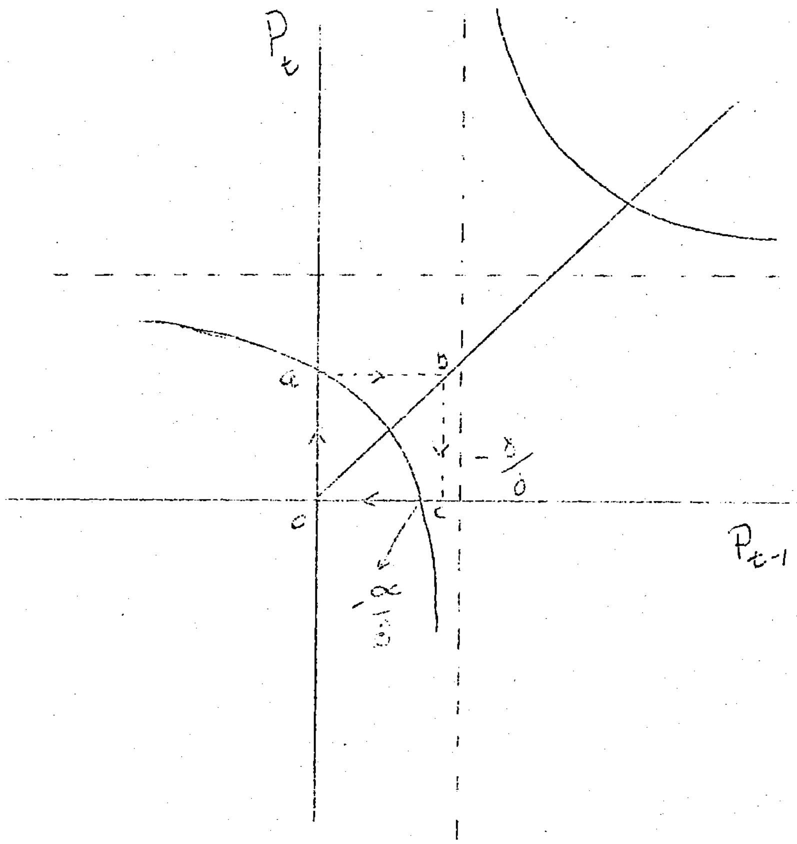


Figure 7

In this case, if P_{t-1} lies in the interval $(\frac{-\gamma}{\delta}, -\frac{\alpha}{\beta})$, the system may oscillate with constant amplitude. A possible cycle is shown by path oabc in Figure 8.



Case (4). $\frac{dP_t}{dP_{t-1}} < 0$ and $\frac{-\gamma}{\delta} = \frac{\beta}{\delta}$

This is a razor's edge case in which both P^*_1 and P^*_2 are points of neutral equilibrium, i.e., any disturbance from these points will lead to oscillation of constant amplitude about these points (Figure 9).

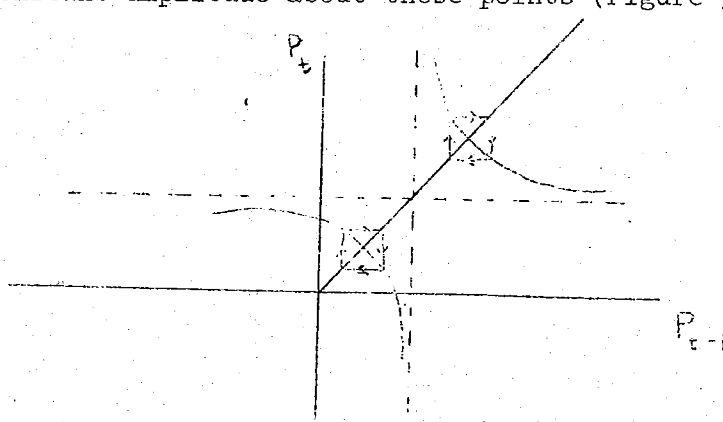


Figure 9

Case (5). $\frac{dP_t}{dP_{t-1}} > 0$ and $\frac{-\gamma}{\delta} > \frac{\beta}{\delta}$

This case is shown in Figure 10, from which it is clear that P^*_1 is a stable equilibrium and P^*_2 is an unstable equilibrium. The behavior of the model is now different from the previous cases since the approach to the equilibrium point is monotonic. P^*_1 and P^*_2 will coincide if $(\gamma - \beta)^2 + 4\alpha = 0$. In that case the system will be stable to the left of P^*_1 but unstable to the right of P^*_1.

Case (6). $\frac{dP_t}{dP_{t-1}} > 0$ and $\frac{-\gamma}{\delta} < \frac{\beta}{\delta}$

This is similar to case (5) except that now P^*_1 is unstable and P^*_2 is stable (see Figure 11).

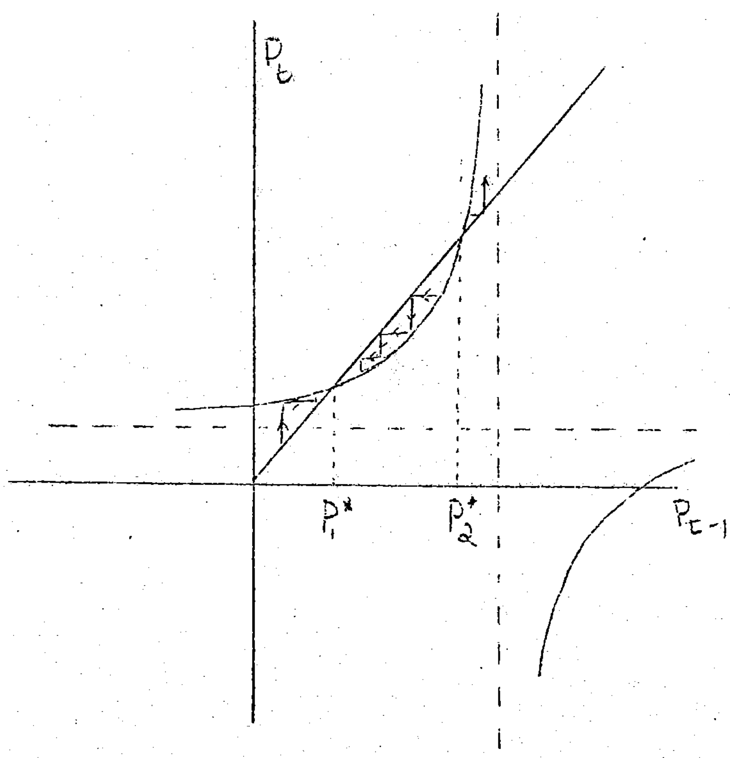


Figure 10

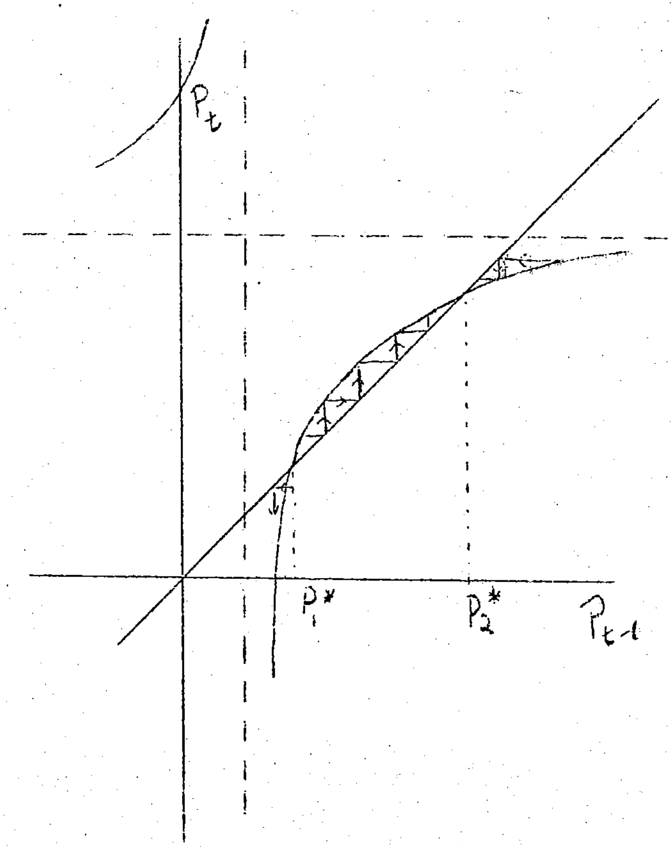


Figure 11

The cases discussed above exhaust all the possibilities with $\beta/\delta > 0$. Therefore, we can conclude that the model will always have a point of stable equilibrium. It is true that for the existence of an equilibrium point we require that $(\gamma - \beta)^2 + 4\,\alpha\,\delta \geq 0$. But this is a less stringent condition than the usual requirement of the simple cobweb model that

$$\left| \frac{\text{slope of the supply curve}}{\text{slope of the demand curve}} \right| < 1.$$

The stability condition no longer depends only on the actual values of the slopes of demand and supply schedules. The crucial assumption that $\beta/\delta < 0$ is quite plausible because both the marginal propensity to consume and the production response to prices are likely to be positive.

It seems that if we have a point of unstable equilibrium, by changing the price, which results in a change in the peasants' income and, hence, in their share of the total consumption, we can move to a point of stable equilibrium. It is clear that the introduction of the income term has made a great difference to the stability of the simple cobweb model. In general, the income effect would result in a nonlinear difference equation which may or may not be more stable than the corresponding linear difference equation, but it is very unlikely that it would have exactly the same conditions for stability.